Exercise 66

Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

Solution

Take the derivative of both sides with respect to x.

$$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2y^2) = 0$$

$$(2x) + (4y) \cdot \frac{dy}{dx} = 0$$

Solve for dy/dx.

$$4y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{4y}$$

$$= -\frac{x}{2y}$$

Set $dy/dx = 1$.

$$1 = -\frac{x}{2y}$$

$$y = -\frac{x}{2}$$

Therefore, the points on the line defined by $y = -x/2$ all have a slope of 1. Find the points that are both on this line and on the ellipse by solving the following system of equations.

$$\begin{cases} x^2 + 2y^2 = 1 \\ y = -\frac{x}{2} \end{cases}$$

Substitute this formula for y into the first equation and solve for x.

$$x^2 + 2\left(-\frac{x}{2}\right)^2 = 1$$

$$x^2 + 2\left(\frac{x^2}{4}\right) = 1$$

$$x^2 + \frac{x^2}{2} = 1$$

$$\frac{3x^2}{2} = 1$$

$$x^2 = \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}}$$

Since $y = -x/2$, the corresponding y-coordinates are

$$x = -\sqrt{\frac{2}{3}}: \quad y = -\frac{1}{2}\left(-\sqrt{\frac{2}{3}}\right) = \frac{1}{\sqrt{6}}$$

$$x = \sqrt{\frac{2}{3}}: \quad y = -\frac{1}{2}\left(\sqrt{\frac{2}{3}}\right) = -\frac{1}{\sqrt{6}},$$

which means the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1 are

$$\left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right) \quad \text{and} \quad \left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right).$$

Below is a graph of the curve and its tangent lines with slope 1.